Cash or nothing derivative

Math 485

December 17, 2013

1 Goal:

To show that the price for a cash or nothing derivative: $V_T = \mathbf{1}_{\{S_T \geq K\}}$ at time t, which is

$$\begin{aligned} V(t, S_t) &= e^{-r(T-t)} N(d_2(t, S_t)), \\ d_2(t, S_t) &= \frac{(r - \frac{1}{2}\sigma^2)(T-t) - \log(\frac{K}{S_t})}{\sigma\sqrt{T-t}} \end{aligned}$$

satisfies the Black-Scholes PDE:

$$\begin{aligned} \frac{\partial}{\partial t}V + \frac{\partial}{\partial x}Vrx + \frac{\partial^2}{\partial x^2}V\sigma^2 x^2 - rV &= 0 \\ V(T, x) = \mathbf{1}_{\{x \geq K\}}. \end{aligned}$$

2 Check the terminal condition:

We want to show that

$$\begin{aligned} V(T, x) &= 1 \text{ if } x \geq K \\ &= 0 \text{ if } x < K. \end{aligned}$$

Indeed if $x > K$ then $\frac{K}{x} < 1$ and $\log(\frac{K}{x}) < 0$. Therefore $d_2(T, x) = \infty$ and $N(d_2(T, x)) = 1$.
Similarly $x \leq K$ then $\frac{K}{x} \geq 1$ and $\log(\frac{K}{x}) \geq 0$. Therefore $d_2(T, x) = -\infty$ and $N(d_2(T, x)) = 0$.

3 Calculations:

1. Derivatives of $d_2(t,x)$:

$$\begin{aligned}
\frac{\partial}{\partial t} d_2(t,x) &= -\frac{r - \frac{1}{2}\sigma^2}{2\sigma\sqrt{T-t}} + \frac{\log(\frac{x}{K})}{2\sigma\sqrt{T-t}^3} \\
&= \frac{1}{2(T-t)}\big[d_2 - \frac{2}{\sigma}(r - \frac{1}{2}\sigma^2)\sqrt{T-t}\big] \\
\frac{\partial}{\partial x} d_2(t,x) &= \frac{1}{x\sigma\sqrt{T-t}} \\
\frac{\partial^2}{\partial x^2} d_2(t,x) &= -\frac{1}{x^2\sigma\sqrt{T-t}}.
\end{aligned}$$

2. Derivatives of V:

$$\begin{aligned}
\frac{\partial}{\partial t} V &= re^{-r(T-t)}N(d_2(t,x)) + e^{-r(T-t)}\phi_z(d_2(t,x))\frac{\partial}{\partial t}d_2(t,x) \\
&= rV + e^{-r(T-t)}\phi_z(d_2(t,x))\frac{\partial}{\partial t}d_2(t,x) \\
\frac{\partial}{\partial x} V &= e^{-r(T-t)}\phi_z(d_2(t,x))\frac{\partial}{\partial x}d_2(t,x) \\
\frac{\partial^2}{\partial x^2} V &= -e^{-r(T-t)}d_2(t,x)\phi_z(d_2(t,x))(\frac{\partial}{\partial x}d_2(t,x))^2 + e^{-r(T-t)}\phi_z(d_2(t,x))\frac{\partial^2}{\partial x^2}d_2(t,x).
\end{aligned}$$

3. Check the cancellations:

a.

$$\begin{aligned}
\frac{\partial}{\partial t} V - rV &= e^{-r(T-t)}\phi_z(d_2)\frac{1}{2(T-t)}\Big[d_2 - \frac{2}{\sigma}(r - \frac{1}{2}\sigma^2)\sqrt{T-t}\Big] \\
&= e^{-r(T-t)}\phi_z(d_2)\frac{1}{2(T-t)}d_2 - e^{-r(T-t)}\phi_z(d_2)\frac{1}{\sigma\sqrt{T-t}}(r - \frac{1}{2}\sigma^2).
\end{aligned}$$

b.

$$\begin{aligned}
\frac{\partial}{\partial x} V rx &= \frac{re^{-r(T-t)}\phi_z(d_2)}{\sigma\sqrt{T-t}} \\
\frac{1}{2}\frac{\partial^2}{\partial x^2} V\sigma^2 x^2 &= \frac{1}{2}\big[-\frac{e^{-r(t-t)}\phi_z(d_2)d_2}{T-t} - \frac{e^{-r(T-t)}\phi_z(d_2)\sigma}{\sqrt{T-t}}\big].
\end{aligned}$$

It is easy to see that everything cancels out now.